Math 4650

Topic 3 – Infinite Series

Def: Suppose we have series $(a_n)_{n=1}^{\infty}$ and we want to make an infinite series out of it.

We define the partial sums by

$$s_k = a_1 + a_2 + a_3 + \dots + a_k$$

for $k \geq 1$.

That is,

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

and so on.

If $\lim_{k\to\infty} s_k$ exists and equals L, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty} a_n = L$.

If $\lim_{k\to\infty} s_k$ does not exist then we say that $\sum_{n=1}^{\infty} a_n$ diverges.

Note: The series can start at other numbers other that $n=1$, such as $\sum_{n=3}^{\infty} a_n = a_3 + a_4 + a_5 + \dots$

In that case just the starting point of S_k.
For example, for above we could do
$S_3 = a_3$, $S_4 = a_3 + a_4$, $S_5 = a_3 + a_4 + a_5$, ...

<u>Ex:</u> Consider $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Let's calculate some partial sums.

k	$S_k = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}$, $k \geq 0$
0	$S_0 = 1$
1	$S_1 = 1 + \frac{1}{2} = 1.5$
2	$S_2 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$
3	$S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$
4	$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375$
$\vdots$	$\vdots$
50	$S_{50} \approx 1.\underbrace{999\ldots99}_{\text{fifteen 9's}}11182$
$\vdots$	$\vdots$
100	$S_{100} \approx 1.\underbrace{999\ldots99}_{\text{thirty 9's}}11139\ldots$

It seems that $\lim_{k\to\infty} S_k = 2$

so, $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$

We will see why this is true, but we will look at $\sum_{n=0}^{\infty} r^n$

We will need this result about sequences.

Theorem: Let $r \in \mathbb{R}$. If $|r|<1$, then $\lim_{n\to\infty} r^n = 0$.

proof: HW $\boxtimes$

Ex: (Geometric series)

We are interested in the series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

Note that

$$\begin{aligned} s_k(1-r) &= (1+r+r^2+\dots+r^k)(1-r) \\ &= 1+r+r^2+\dots+r^k \\ &\quad -r-r^2-\dots-r^k-r^{k+1} \\ &= 1-r^{k+1} \end{aligned}$$

Thus,

$$s_k = \frac{1-r^{k+1}}{1-r}$$

Thus, if $|r|<1$ then

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r} = \frac{1-0}{1-r} = \frac{1}{1-r}$$

Therefore, if $|r|<1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Ex: $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

$r = 1/2,\ |r|<1$

Ex: Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$$

We can use partial fractions to get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$1 = A(n+1) + Bn$

$\underline{n=-1:}$ $1 = A(0) + B(-1)$

$B = -1$

$\underline{n=0:}$ $1 = A(1) + B(0)$

$1 = A$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and

$$s_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2}$$

$$s_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - 1/3$$

$$s_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

In general for $k \geq 1$ we get

$$s_k = 1 - \frac{1}{k+1}$$

Thus,

$$\lim_{k \to \infty} s_k = 1 - 0 = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

This series is called a "telescoping series" because of how the terms cancel each other out in the partial sums.

Theorem: (Divergence Test)

If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.

[Thus, if $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges]

Proof:

Suppose $\sum a_n$ converges to L.

Then $\lim\limits_{k\to\infty} s_k = L$.

Since

$$a_n = (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1})$$
$$= s_n - s_{n-1}$$

We get that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$$

$\blacksquare$

Ex: $\sum\limits_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

diverges because $\lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0}$

$$= 1 \neq 0$$

Ex: The harmonic series is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Note that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

So we can't use the divergence test.

However, it turns out that this series will diverge.

Let's consider the partial sums

$$s_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

In order for $\lim\limits_{k\to\infty} s_k$ to exist we would need that (s_k) is a Cauchy sequence.

We will show this is not the case.

If $m > n$, then

$$\begin{aligned} s_m - s_n &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \\ &> \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m} \\ &= \frac{m-n}{m} \\ &= 1 - \frac{n}{m} \end{aligned}$$

In particular, if $m = 2n$ and $n \geq 3$ then

$$s_{2n} - s_n = 1 - \frac{1}{n} > \frac{1}{2}$$

This implies that $(s_k)_{k=1}^{\infty}$ is not Cauchy.

Why?

If (s_k) was cauchy then there would exist $N \geq 3$ where if $m, n \geq N$, then

$$|s_m - s_n| < \frac{1}{2}$$

But picking $n \geq N \geq 3$ and $m = 2n \geq N$ we get $|s_m - s_n| = |s_{2n} - s_n| > \frac{1}{2}$.

Thus, (s_k) diverges and so does $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\blacksquare$

Ex: (p-series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p>1.$$

pf: Since the terms of the series are all positive, the partial sums (s_k) form a monotonically increasing sequence. Thus, it suffices to prove that there is some subsequence of (s_k) that is bounded and this bound will bound all the terms of (s_k) which will prove that (s_k) converges by the monotone convergence theorem.

Why? Exercise

Set $k_1 = 2^1 - 1 = 1$.

Then, $s_{k_1} = s_1 = \frac{1}{1^p} = 1$.

Set $k_2 = 2^2 - 1 = 3$.

Then,

$$s_{k_2} = s_3 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right)$$

$\boxed{2^p < 3^p \text{ since } p>1}$

$$= 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

Set $k_3 = 2^3 - 1 = 7$.

Then,

$$s_{k_3} = s_7 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2$$

In general, if $k_j = 2^j - 1$ and $r = \frac{1}{2^{p-1}}$, then

$$0 < s_{k_j} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^{j-1}$$

$$= 1 + r + r^2 + \dots + \dots + r^{j-1}$$

$$= \frac{1 - r^j}{1 - r}$$

$$< \frac{1}{1 - r}$$

Thus, we have a bounded subsequence $(s_{k_j})_{j=1}^{\infty}$

As described above we get that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

$\blacksquare$

Note: It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$ where B_{2k} is the 2k-th Bernoulli number

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020569\ldots \leftarrow \text{Apéry's constant}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278\ldots$$

(These are values of the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$)

Theorem: (Comparison test)

Suppose that $0 \leq a_n \leq b_n$ for all $n \geq K$ for some fixed $K > 0$.

Then:

① If $\sum b_n$ converges, then $\sum a_n$ converges.

② If $\sum a_n$ diverges, then $\sum b_n$ diverges.

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proof:

② is the contrapositive of ①, so we just need to prove ①.

Suppose that $\sum b_n$ converges.

Let (s_k) be the partial sums of $\sum b_n$ and (t_k) be the partial sums of $\sum a_n$.

Since $\sum b_n$ converges we know that (s_k) converges, so it's a Cauchy sequence.

Let $\varepsilon > 0$.

Then there exists $N > K$ where if $m \geq n \geq N$, then $|s_m - s_n| < \varepsilon$.

Therefore if $m \geq n \geq N$, then

$$
\begin{aligned}
|t_m - t_n| &= |a_1 + a_2 + \cdots + a_m - a_1 - a_2 - \cdots - a_n| \\
&= |a_{n+1} + a_{n+2} + \cdots + a_m| \\
&= a_{n+1} + a_{n+2} + \cdots + a_m \\
&\leq b_{n+1} + b_{n+2} + \cdots + b_m \\
&= |b_{n+1} + b_{n+2} + \cdots + b_m| \\
&= |b_1 + b_2 + \cdots + b_m - b_1 - b_2 - \cdots - b_n| \\
&= |s_m - s_n| \\
&< \varepsilon
\end{aligned}
$$

Thus, (t_k) is a Cauchy sequence.

So, (t_k) converges and so does $\sum a_n$. $\blacksquare$

<u>Ex:</u> (p-series)

Suppose $0 < p < 1$.

If $n \in \mathbb{N}$, then $n^p \leq n$.

So, if $n \in \mathbb{N}$, then $\frac{1}{n} \leq \frac{1}{n^p}$.

Thus, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test we know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Ex: Since $0 < \frac{1}{n^2+n} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the comparison test we get that $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Theorem: (Alternating series test)

Let (a_n) be a monotonically decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$.

Then, the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

converges.

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proof: Let (s_k) be the partial sums. First we look at the subsequence (s_{2k}).

Since (a_n) is monotonically decreasing for all $n \geq 1$, we know that $a_n - a_{n+1} \geq 0$

Since

$$s_{2k} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2k-1} - a_{2k})$$

we get that $(s_{2k})_{k=1}^{\infty}$ is a monotonically increasing sequence.

Also,

$$s_{2k} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2k-2} - a_{2k-1}) - a_{2k}$$
$$\leq a_1$$

Thus, since $(s_{2k})_{k=1}^{\infty}$ is monotonically increasing and bounded from above, by the monotone convergence theorem $\lim_{k\to\infty} s_{2k} = L$ for some $L \in \mathbb{R}$.

We now show that $\lim_{k\to\infty} s_k = L$.

Let $\varepsilon > 0$

Since $\lim_{k\to\infty} s_{2k} = L$ there exists $N_1 > 0$ where if $n \geq N_1$ then $|s_{2n} - L| < \frac{\varepsilon}{2}$

Since $\lim_{k\to\infty} a_k = 0$, there exists $N_2 > 0$ where if $n \geq N_2$ then $|a_{2n+1}| = |a_{2n+1} - 0| < \frac{\varepsilon}{2}$.

If $n \geq \max\{N_1, N_2\}$ then

$$
\begin{aligned}
|s_{2n+1} - L| &= |s_{2n} + a_{2n+1} - L| \\
&\leq |s_{2n} - L| + |a_{2n+1}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, for $n \geq \max\{N_1, N_2\}$ we get both $|s_{2n} - L| < \frac{\varepsilon}{2} < \varepsilon$ and $|s_{2n+1} - L| < \varepsilon$

This implies that $|s_k - L| < \varepsilon$ for all $m \geq 2\max\{N_1, N_2\}$.

So, $(s_k)_{k=1}^{\infty}$ converges to L. $\blacksquare$

<u>Ex:</u> Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $0 < \frac{1}{n+1} < \frac{1}{n}$ for all $n \geq 1$ we get that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges.
